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GENERALIZED FROBENIUS GROUPS. II

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ABSTRACT

A pair (G, K) in which G is a finite group and K a normal nontrivial proper subgroup of G is said to be an F2-pair (a Frobenius type pair) if $|C_G(x)| = |C_{G/K}(xK)|$ for all $x \in G \setminus K$. A theorem of Camina asserts that in this case either K or G/K is a p-group or else G is a Frobenius group with Frobenius kernel K. The structure of G will be described here under certain assumptions on the Sylow p-subgroups of G.

Introduction

A pair (G, K), where G is a finite group and K a normal nontrivial subgroup of G, is said to satisfy condition F2 *if* $|C_{G/K}(xK)| = |C_G(x)|$ *for all* $x \in G \setminus K$. *Such a pair will be also called an* F2-*pair*. We know of 5 types of examples of such groups; these will be described below. Our purpose here is to show that under certain conditions these are the only examples.

Here are the 5 types of examples:

Type 1: *G is a Frobenius group and K is the Frobenius kernel.*

Type 2: F2-*pairs (G, K), where G is a p-group*. These can be found in [M, M1]. They exist for every prime p. The simplest example here is G being an extra-special p-group and $K = Z(G)$.

Type 3: F2-*pairs, (G, K) in which $K < P < G$ where P is a normal Sylow p-subgroup of G. Here (P, K) is also an* F2-*pair*. *Furthermore $G = RP$ where R*

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is a p-complement and RK is a Frobenius group with Frobenius kernel K and Frobenius complement R. Examples of this type can be found in [CM]; they exist for any prime p.

Type 4: *G is a Frobenius group in which a Frobenius complement is isomorphic to* Q_8 (*the quaternion group of order* 8) *and K is a subgroup of index* 4. Examples of this type can be found in [C] and [CM].

Type 5: Two special examples of order $2^a 3^b$, one with K a 2-group and one with K a 3-group, that will be discussed later.

Many properties of F2-pairs of type 2 are established in [M, M1, Ma] and of those of type 3 in [CM]. Recently, an example of class 4 was constructed by C. K. Gupta.

A Theorem of Camina, [C], states that if (G, K) is an F2-pair not of type 1, then either K or G/K is a p-group for some prime p. Hence, there is a prime associated with every F2-pair not of type 1. To specify this prime, *we will call an* F2-*pair*, (G, K), *an* F2(p)-*pair if either K or G/K is a p-group, but* (G, K) *is not of type* 1. We note that in F2-pairs (G, K) of types 3 and 5 we have that K is a p-group, while in those of types 2 and 4 G/K is a p-group.

Let (G, K) be an F2(p)-pair with K a p-group and $P \in \mathrm{Syl}_p(G)$. We will see that K is a member of every central series of P. Thus the cases $K = P' = [P, P]$ and $K = Z(P)$ are of interest. It turns out that in both cases G has to be of type 3.

THEOREM 1. *Let* (G, K) *be an* F2(p)-*pair for some prime p and let* $P \in \mathrm{Syl}_p(G)$. *Suppose that* either (i) $K = Z(P)$ or (ii) $K = P'$. *Then* (G, K) *is of type* 3.

COROLLARY 2. *Let* (G, K) *be an* F2(p)-*pair for some prime p with K a p-group. Let* $P \in \mathrm{Syl}_p(G)$. *Suppose that either*

(i) *P/K is abelian, or*

(ii) *The nilpotency class of P is at most* 3, *or*

(iii) $|P| \leqq p^5$.

Then (G, K) *is of type* 3.

REMARK. The examples of type 5 are F2(p)-pairs in which K is a p-group but (G, K) is not of type 3. In these examples the nilpotency class of P is 4 and $|P| = p^6$. Thus the assumptions on the nilpotency class or the order of P in Corollary 2 cannot, in general, be relaxed.

For F2(p)-pairs, (G, K), with G/K a p-group we have:

THEOREM 3. *Let (G, K) be an* F2(p)*-pair with G/K a p-group and let $P \in \mathrm{Syl}_p(G)$. Assume that* either one *of the following holds*:

(1) *P is regular (in the sense of P. Hall).*
(2) *p is odd and P is abelian by cyclic.*
(3) *P is of maximal class.*
(4) *The nilpotency class of P is at most $p + 1$.*
(5) $K \cap P = Z(P)$.
(6) *P contains an abelian subgroup of index p^2.*
(7) *The nilpotency class of P is at most* 4.

Then (G, K) is either of type 2 *or of type* 4.

REMARK. We know of no other examples of F2(p)-pairs in which G/K is a p-group (except for pairs of types 2 and 4). Also, in general $K \cap P$ is a member of both the lower and the upper central series of P and G has a normal p-complement (see Lemma 2.1).

Section 1 of the paper includes the proofs of Theorem 1, Corollary 2, some other properties of F2(p)-pairs (G, K) with K a p-group and a discussion of pairs of type 5. Pairs with G/K a p-group are considered in Section 2 in which the proof of Theorem 3, as well as some other properties of such pairs, can be found.

Our notation is standard. We will mention here three pieces of notation: The nilpotency class of the nilpotent group T will be denoted by cl(T) and the lower central series of the group S will be denoted by $S_1 \geqq S_2 \geqq \cdots \geqq S_m$, namely $S = S_1$, $S_2 = [S, S] = S'$, $S_i = [S_{i-1}, S]$ for $1 \leqq i \leqq m$. The i-th term of the upper central series of S will be denoted by $Z_i(S)$.

We note here that (G, K) is an F2-pair if and only if for each $g \in G \setminus K$ and each $h \in K$, g is conjugate in G to gh (see [CM, 3.1]).

I. F2(p)-pairs, (G, K), with K a p-group

PROPOSITION 1.1. *Let (G, K) be an* F2(p)*-pair with K a p-group and let $P \in \mathrm{Syl}_p(G)$. Then K appears as a term in every central series of P.*

PROOF. Let $P = Q_0 \geqq Q_1 \geqq Q_2 \geqq \cdots \geqq Q_{c+1} = 1$ be a central series of P (not necessarily the lower central series). As $Q_c \leqq Z(P)$ we get by [CM, 3.4] that $Q_c \leqq K$. So, there is an index i such that $1 \neq Q_i \leqq K$ but $Q_{i-1} \nleqq K$. We claim that $Q_i = K$. Assume the contrary, namely, that $Q_i < K$ and let $x \in Q_{i-1} \setminus K$. We have that $[x, P] \leqq Q_i$. As the number of conjugates of x in P is equal to the number of elements in the set $\{[x, y] \mid y \in P\}$, we get that

$|P:C_P(x)| \leqq |Q_i| < |K|$. It follows that $|P/K| < |C_P(x)|$. Now the F2 property implies that:

$$|G/K|_p = |P/K| \geqq |C_{G/K}(xK)|_p = |C_G(x)|_p \geqq |C_P(x)| > |P/K| = |G/K|_p,$$

a contradiction. ∎

In general, if (G, K) is an F2(p)-pair, then $P \cap K$ is a member of every central series of P. The proof is as above or from [CM, M1].

PROOF OF THEOREM 1. It suffices to show that $P \lhd G$. Once $P \lhd G$, then $G = PM$ where M is a p-complement and we get that (G, K) is of type 3 by [CM, 4.2 and 4.3].

(i) $Z(P) = K \lhd G$ implies that $K = Z(P^g)$ for all $g \in G$ and by [CM, 4.5] we get that $P = P^g$ for all $g \in G$ so that $P \lhd G$ as desired.

(ii) Let G be a minimal counterexample and let $N \neq 1$ be a minimal normal subgroup of G contained in K. If $N < K = P'$, then $(G/N, K/N)$ is an F2(p)-pair. As $P/N \in \mathrm{Syl}_p(G/N)$ and since $(P/N)' = P'/N = K/N$ induction implies that $P/N \lhd G/N$, a contradiction. Thus $K = N$ is a minimal normal subgroup of G. By [CM, 3.4 and 3.5] we get that $\mathrm{cl}(P) > 1$ and that $Z(Q) \leqq K$ for all $Q \in \mathrm{Syl}_p(G)$. As K is minimal normal we get that $K = \langle Z(Q) \mid Q \in \mathrm{Syl}_p(G) \rangle$ and consequently K centralizes $\bigcap\{Q \mid Q \in \mathrm{Syl}_p(G)\} = O_p(G)$. Hence, $O_p(G) \leqq C_G(K) \lhd G$. By [CM, 4.3] $C_G(K)$ is a p-group and so $C_G(K) = C_P(K) = O_p(G)$. If $\mathrm{cl}(P) = 2$ then by Proposition 1.1 $K = Z(P)$ and we get a contradiction using (i). Thus $\mathrm{cl}(P) \geqq 3$.

We now break the proof into steps.

Step 1. (P, K) is an F2-pair. In particular p is odd.

Let $x \in P \setminus P'$ and set $A = \{[x, y] \mid y \in P\}$. By the remark at the end of the introduction we have to show that $A = P'$. Clearly, $A \leqq P'$ and $|A| = |\{y^{-1}xy \mid y \in P\}| = |P:C_P(x)|$. Now, P/P' centralizes xP' and as $P/P' \in \mathrm{Syl}_p(G/P')$ we obtain that $|P/P'| = |C_{G/P'}(xP')|_p$. The F2 property implies that

$$|C_P(x)| \leqq |C_G(x)|_p = |C_{G/P'}(xP')|_p = |P:P'|.$$

Consequently $|A| = |P:C_P(x)| \geqq |P'|$ so that $A = P'$ as desired. The fact that p is odd now follows from [M1, 3.1].

Step 2. $O_p(G)$ is abelian.

As $K = P_2$ and $O_P(G) = C_P(P_2)$ we use [H, III. 2.14] to obtain that

$$(O_p(G))' = [C_P(P_2), C_P(P_2)] \leqq Z(P).$$

But $Z(P) < K$ and hence $(O_p(G))' < K$. Since K is a minimal normal subgroup we get that $(O_p(G))' = 1$ so that $O_p(G)$ is abelian.

Step 3. $O_p(G) > K$.

Suppose that $O_p(G) = K$. Then [CM, 4.4] implies that $O_p(G/K) = O_{p'}(G/K) = 1$. If, now, $\bar{M}$ is a minimal normal subgroup of G/K, then $\bar{M}$ has to be a direct product of nonabelian simple groups. So $|\bar{M}|$ is even and, as $p \neq 2$, the Sylow 2-subgroups of $\bar{M}$ are either cyclic or generalized quaternion (see [CM, 4.3]). This contradicts [BS] and [H, IV.2.8]. Thus $O_p(G) > K$.

Step 4. Set $|P_2 : P_3| = p^n$, then $|P : P_2| = p^{2n}$.

Note that $[P/P_4, P/P_4] = P_2/P_4$ and $[P/P_4, P/P_4, P/P_4] = P_3/P_4$. As (P, K) is an F2-pair it follows that $(P/P_4, K/P_4)$ is an F2-pair with $K/P_4 = [P/P_4, P/P_4]$. Now $\mathrm{cl}(P/P_4) = 3$ and $|P_2/P_4 : P_3/P_4| = p^n$ so we can apply [M, 5.2] to conclude that $|P : P_2| = |P/P_4 : P_2/P_4| = p^{2n}$.

Step 5. Final contradiction.

By Step 3 there is an element $x \in O_p(G) \setminus K$. As $O_p(G)$ is abelian and (P, K) is an F2-pair (by steps 1 and 2) we have that

$$|O_p(G)| \leqq |C_P(x)| = |C_{P/K}(xK)| = |C_{P/P'}(xP')| = |P : P'| = p^{2n}.$$

Next, the Sylow p-subgroups of G/K are abelian and by [B] there exist two Sylow subgroups R and Q of G such that

$$O_p(G/K) = O_p(G)/K = (R/O_p(G)) \cap (Q/O_p(G)).$$

Thus $O_p(G) = R \cap Q$. Note that $K = P' = R' = Q'$. Let $1 \neq z \in Z(R) \subseteq K$ (by [CM, 3.4]). As no p'-element of G centralizes an element of K (see [CM, 4.3]) we get that $C_G(z)$ is a p-group and consequently $C_G(z) = R$. It follows that $C_Q(z) = R \cap Q = O_p(G)$. We note that as $z \in K = Q' = Q_2$ we know that $zQ_3 \in Q_2/Q_3 \subseteq Z(Q/Q_3)$. Finally we use [I, 2.24] and step 4 to obtain:

$$|O_p(G)| = |C_Q(z)| \geqq |C_{Q/Q_3}(zQ_3)| = |Q/Q_3| = |P/P_3| = p^{3n}.$$

This contradicts the conclusion of the previous paragraph. ■

Proof of Corollary 2. (i) As P/K is abelian, $P' \subseteq K$. By Proposition 1.1 $K \subseteq P'$, hence $K = P'$ and we are done by Theorem 1.

(ii) Proposition 1.1 implies that either $K = P'$ or $K = P_3$. In the former case the result follows from Theorem 1. If $K = P_3$ then $\mathrm{cl}(P) = 3$ so that $K \subseteq Z(P)$. Again, by Proposition 1.1 we get that $Z(P) \subseteq K$ so that $K = Z(P)$ and the result follows from Theorem 1.

(iii) Let G be a counterexample of minimal order. Since the assumptions are inductive modulo normal subgroups contained in K we get that K is a minimal normal subgroup. As in the proof of Theorem 1 we get that it is suffices to prove that $P \lhd G$. By Part (ii) $\mathrm{cl}(P) = 4$ so that $|P| = p^5$. So P has a unique normal subgroup of each of the orders p, p^2, p^3 (see [H, III. 14.2]). By Theorem 1 $K \neq Z(P) = P_4$ and $K \neq P_2$ so that Proposition 1.1 implies that $K = P_3 = Z_2(P)$. Note that $|Z(P)| = |P_4| = p$, $|K| = p^2$ and $|P_2| = |Z_3(P)| = p^3$.

As in the proof of Theorem 1(ii) we get that $K = \langle Z(Q) \mid Q \in \mathrm{Syl}_p(G) \rangle$ and that $O_p(G) = C_G(K)$. Also, any two distinct Sylow p-subgroups of G have disjoint centers ([CM, 4.5]) and as K has exactly $p+1$ subgroups of order $|Z(P)| = p$ we conclude that G has exactly $p+1$ Sylow p-subgroups. Let Ω be the collection of these $p+1$ subgroups and let N be the kernel of the action of G on Ω. Clearly the Sylow p-subgroup of N is $P \cap N = O_p(G)$. As G/N is a permutation group of degree $p+1$ on Ω it follows that $p = |G/N|_p = |PN/N| = |P/O_p(G)|$ so that $|O_p(G)| = p^4$.

If $O_p(G)$ is abelian an element $x \in O_p(G) \setminus K$ would have to satisfy

$$p^4 = |O_p(G)| \leqq |C_G(x)|_p = |C_{G/K}(xK)|_p \leqq |G/K|_p = p^3$$

(by the F2-property), a contradiction. Thus $O_p(G)$ is not abelian and as K is minimal normal in G we get that

$$[O_p(G), O_P(G)] = K, \qquad [O_p(G), O_p(G), O_p(G)] = 1$$

and both K and $O_p(G)/K$ are elementary abelian (see [H, III.7.1]). In particular, $O_p(G)$ is of class 2. It follows that the Frattini subgroup of $O_p(G)$ is also K and so $O_p(G)$ is generated by two elements. By [H, III. 1.11(c)] $K = [O_p(G), O_p(G)]$ is cyclic, a contradiction. ∎

EXAMPLES. As mentioned before, pairs (G, K) of type 5 are examples of F2(p)-pairs with K a p-group (for $p = 2$ or 3), which are not of type 3. This fact is proved in [G, p. 383] where these examples were introduced. Since normality of P implies that the pair is of type 3, it was asked in [CM] whether for any F2(p)-pair with K a p-group, (P, K) is also a F2-pair. We will see now that the Gagola's examples (type 5) are counterexamples to this assertion. These examples do not seem to be generalizable for $p > 3$, so for $p > 3$ it is still possible that $P \lhd G$ for F2(p)-pairs, (G, K), with K a p-group.

Let p be the prime 2 or 3, $R = Z/p^2Z$, $M = R \oplus R$ and $K = \Omega_1(M)$.

Consider the group S, whose elements are 2×2 matrices with coefficients in R, as follows: If $p = 2$,

$$S = \left\langle \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} \right\rangle$$

and if $p = 3$,

$$S = \left\langle \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 4 \\ 6 & 7 \end{pmatrix} \right\rangle.$$

More detailed description of the structure on S can be found in [G].

Consider now, in both cases, the (natural) semidirect product, G, of M with S. In [G, p. 367 and 383] it is shown that (G, K) is an F2(p)-pair. Let $P \in \mathrm{Syl}_p(G)$, then P is not normal in G. We show now that (P, K) is not an F2-pair. For $p = 2$, P is the semidirect product of M with $\langle a \rangle$ where

$$a = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}.$$

Note that P/K is not abelian, since aK interchanges, by conjugation in P/K, xK and yK, where $x = (0, 1) \in M$ and $y = (1, 1) \in M$. Thus $xK \notin Z(P/K)$ and $|C_{P/K}(xK)| \leqq 8$. But $|C_P(x)| \geqq 16$, since M centralizes x.

For $p = 3$, P is generated by M and

$$a = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 4 \\ 6 & 7 \end{pmatrix}.$$

Again, P/K is not abelian: bK permutes cyclically, by conjugation in P/K, the three elements xK, yK, zK, where $x = (0, 1)$, $y = (1, 1)$, $z = (-1, 1)$, $x, y, z \in M$. Now, $|C_{P/K}(xK)| \leqq 27$, while $|C_P(x)| \geqq 81$. Hence (P, K) is not an F2-pair in either case.

II. (F2(p))-pairs, (G, K), with G/K a p-group

We start with a definition.

Definition. A finite group X is called here a Frobenius–Wielandt complement (FW-complement for short) if there exist a finite group G with $H \lhd G$ and a prime q not dividing $|G|$ such that:

(i) G acts on an elementary abelian q-group Q.

(ii) All the elements in $G \setminus H$ act fixed point freely on Q.

(iii) $X \cong G/H$.

See [E, p. 564] for its definition of FW-complements and the second part of Theorem 1.5 of [E] (with its short proof) for the alternate definition which is like ours.

LEMMA 2.1. *Let (G, K) be an* F2(p)*-pair with G/K a p-group Then*:

(a) *G has a normal p-complement.*

(b) *Assume that G is not a p-group. Let $P \in \mathrm{Syl}_p(G)$ and R the normal p-complement of G. Then all the elements of $G \setminus K$ act fixed point freely on R. In particular $T/T \cap K$ is a FW-complement for every subgroup T of P with $T \nsubseteq K$.*

(c) *G is solvable.*

PROOF. Write $G = PK$ where P is a Sylow p-subgroup of G. By [CM, 3.6] $(P, P \cap K)$ is an F2-pair. By [M, 2.1] $P \cap K \subseteq [P, P] \subseteq \Phi(P)$ where $\Phi(P)$ is the Frattini subgroup of P. By a Theorem of Tate [H, IV.4.7] K has a normal p-complement which is clearly a normal p-complement of G. This proves (a).

To see (b), take $g \in G \setminus K$. Then $|C_G(g)| = |C_{G/K}(gK)|$ which is a p-power. Hence g commutes with no p'-element. Therefore all elements of $G \setminus K$, including those in $T \setminus T \cap K$, act fixed-point-freely on R. By the Frattini argument, P normalizes some Sylow q-subgroup, Q_0, of R. Set $Q = \Omega_1(Z(Q_0))$. Then all elements of $P \setminus P \cap K$ (and those of $T \setminus T \cap K$) act fixed-point-freely on Q and (b) follows. Part (c) is a consequence of the classification of the finite simple groups (see [GO1, p. 55]) and parts (a) and (b). ■

REMARK. Lemma 2.1(a) was also proved, independently, by Isaacs [I1, Th.C].

LEMMA 2.2. *Let (P, M) be an* F2*-pair, where P is a p-group of class c. If $M \neq Z(P)$ then* $\mathrm{cl}(Z_{c-1}(P)) \leqq c - 2$.

PROOF. Set $Z_j = Z_j(P)$ for $1 \leqq j \leqq c$. By [M, 2.1] $M = Z_i = P_{c+1-i}$ for some i. By assumption $i \geqq 2$ and so by [M, 2.4] $\mathrm{cl}(P) \geqq 3$. Now [H, III.2.11] implies that

$$[Z_i, Z_{c-1}] = [P_{c+1-i}, Z_{c-1}] \subseteq Z_{c-1-(c+1-i)} = Z_{i-2}.$$

Therefore

$$1 \leqq Z_1 \leqq Z_2 \leqq \cdots \leqq Z_{i-2} \leqq Z_i \leqq Z_{i+1} \leqq \cdots \leqq Z_{c-1}$$

is a central series of length $c - 2$ for Z_{c-1}. ■

PROOF OF THEOREM 3, PARTS 1–6. Let G be a counterexample of minimal order. Set $M = P \cap K$, then G is not a p-group and $M \neq 1$ (see [CM, 3.3]). By Lemma 2.1 P/M is a FW-complement and by [C, Lemma 7] P/M is not cyclic.

Also, as $G = KP$, (P, M) is an F2-pair (see [CM, 3.6]). Furthermore, by [CM, 5.1], $\mathrm{cl}(P) \geqq 3$. We claim that P is a maximal subgroup of G. For if $P < U < G$ for some subgroup U of G then $G = UK$ and so $(U, K \cap U)$ is an F2-pair (by [CM, 3.6]) with $U/K \cap U \cong UK/K \cong G/K$. Now, $K \cap U \supseteq K \cap P \neq 1$, so that $K \cap U$ is not a p'-group and so U is not a Frobenius Group with Frobenius kernel $K \cap U$ (see [CM, 3.2]). Hence, $(U, K \cap U)$ is an F2(p)-pair with $U/U \cap K$ p-group. By induction we get that $|P| = 8$ so that $\mathrm{cl}(P) < 3$, a contradiction. Hence, P is maximal in G.

Set $Z = Z(P)$, let $z \in Z$ and write $D = C_G(z)$. Then $D \geqq P$. Suppose that $D = G$. By [CM, 3.4], $z \in K$ and so $(G/\langle z \rangle, K/\langle z \rangle)$ is an F2-pair. If $G/\langle z \rangle$ is a Frobenius group with Frobenius Kernel $K/\langle z \rangle$ then G/K is isomorphic to a Frobenius complement forcing it to be either cyclic or generalized quaternion. This contradicts Lemma 7 of [C]. Thus $(G/\langle z \rangle, K/\langle z \rangle)$ is an F2(p)-pair and induction implies that $p = 2$ and $P/\langle z \rangle \cong Q_8$ with

$$|P:M| = |P:P \cap K| = |G:K| = |G/\langle z \rangle : K/\langle z \rangle| = 4.$$

As (P, M) is an F2-pair, $P' \supseteq M$ (by [M, 2.1]) and hence $M = P'$. Then [M1, 3.1] implies that $\mathrm{cl}(P) = 2$, a contradiction. It follows that $D < G$ and as P is maximal we have that $D = P$. We conclude that Z acts fixed-point-freely on the normal p-complement, Q, whose existence follows from Lemma 2.1. Then Z is a Frobenius complement in the Frobenius group ZQ. In particular Z is cyclic. Also, Q as a Frobenius kernel is nilpotent.

We will reach a contradiction in several cases that will cover all the cases of parts 1–6 of the Theorem.

CASE 1: *P is regular*. We get a contradiction by [LP].

CASE 2: *p is odd and P is either of maximal class or abelian by cyclic*. Using [S] we get that $M \supseteq [P, P]$. Now, (P, M) is an F2-pair and so [M, 2.1] implies that $M = [P, P] = P'$. We claim that P/M is of rank 2. This is clearly true when G is of maximal class. If P is abelian by cyclic, let A be a normal abelian subgroup of P with P/A cyclic. Then $P' \subseteq A$ and $M = P' < A$ because otherwise P/M would be cyclic. As $M = P \cap K = A \cap K$ we get from Lemma 2.1 that A/M is an FW-complement. As A is abelian, A/M must be cyclic (see for example [LP] and hence P/M is metacyclic, which is of rank 2, as claimed. By [M, 2.1 and 2.3] P/M is elementary abelian and so $|P/M| = p^2$. Note that $(P/P_4, M/P_4)$ is an F2-pair with $\mathrm{cl}(P/P_4) = 3$ and $M/P_4 = [P/P_4, P/P_4]$. By [M, 5.2] we have that $|P/P_4 : M/P_4| \geqq p^4$, a contradiction.

CASE 3: *P has a cyclic subgroup of index 2*. Let S be such a subgroup. As

P/M is not cyclic, $S \neq M$. Pick $x \in S \setminus M$. Then the F2 property of (P, M) implies that

$$|P/M| \geqq |C_{P/M}(xM)| = |C_P(x)| \geqq |S|$$

so that $|M| \leqq |P/S| = 2$. It follows that $|M| = 2$ and $|P/M| = |C_{P/M}(xM)|$. Then $S/M \subseteq Z(P/M)$ which implies that P/M is abelian. Clearly $M \subseteq Z(P)$ and by [M, 2.1] $M = Z(P)$. Hence, $\mathrm{cl}(P) = 2$, a contradiction.

CASE 4: $M = Z(P)$. Set $Z = Z(P)$. Then Z is elementary abelian by [M, 2.2], and as Z is cyclic, $|Z| = p$. If there is an element of order p in $P \setminus M$ then [C, Lemma 4] implies that K is nilpotent. This is impossible as Z acts fixed-point-freely on Q and $Z < K = MQ$. Hence all the elements of order p of P lie in M, which has order p. Thus, P has exactly one subgroup of order p and P is not cyclic. Therefore P is generalized quaternion and a contradiction is obtained by the previous case.

Using the cases so far we can assume that $\mathrm{cl}(P) \geqq 3$, $M \neq Z(P)$ and P is neither regular nor of maximal class (the latter follows from cases 2 and 3 and [GO, pp. 191–194]).

CASE 5: $\mathrm{cl}(P) \leqq p + 1$ *and* $M \neq P'$. Set $c = \mathrm{cl}(P)$ and $Z_j = Z_j(P)$ for $1 \leqq j \leqq c$. By [M, 2.1] $M = Z_i$ for some $i \geqq 1$. As $M \neq Z(P)$, $i > 1$. If $i = c - 1$, then $M = P'$ by [M, 2.1], a contradiction. Thus $1 < i \leqq c - 2$ (so $\mathrm{cl}(P) > 3$ in this case, for if $\mathrm{cl}(P) = 3$ then $M = P'$ or $Z(P)$). By Lemma 2.2 we get that $\mathrm{cl}(Z_{c-1}) \leqq c - 2 \leqq p + 1 - 2 < p$ so that Z_{c-1} is regular (see [H, III.10.2]). Lemma 2.1(b) implies that Z_{c-1}/M is a FW-complement. It follows from [LP] that $Z_{c-1}/M = Z_{c-1}/Z_i$ is cyclic.

Suppose first that $i \leqq c - 3$. Then $Z_2(P/M) = Z_2(P/Z_i) = Z_{i+2}/Z_i \subseteq Z_{c-1}/Z_i$ and so $Z_2(P/M)$ is cyclic. As P/M is not cyclic we get $\mathrm{cl}(P/M) > 2$ and that $p = 2$ and P/M contains a cyclic subgroup of index 2 (see [H, III. 7.7]). Now [GO, pp. 191–193] implies that P/M is of maximal class and that $|P/M : (P/M)'| = 4$. By [M, 2.1] we have that $M = Z_i = P_{c+1-i} \subseteq P_2 = P'$ and consequently $(P/M)' = P'/M$ which implies that $|P : P'| = 4$. Then P itself is of maximal class ([GO, p. 194]), a contradiction.

Hence $i = c - 2$. Set $Q = P/M$. Then $Z(Q) = Z(P/Z_{c-2}) = Z_{c-1}/Z_{c-2}$ is cyclic. Set $X = P/Z_{c-3}$ and $Y = M/Z_{c-3}$. Then (X, Y) is an F2-pair in which $Y = Z_{c-2}/Z_{c-3} = Z(X)$. By [M, 2.2] we know that $W = Z_2(X)/Z(X)$ and $V = Z_3(X)/Z_2(X)$ are elementary abelian. Note that:

$$W = \frac{Z_{c-1}/Z_{c-3}}{Z_{c-2}/Z_{c-3}} \cong Z_{c-1}/Z_{c-2} = Z(Q)$$

and

$$V = \frac{Z_c/Z_{c-3}}{Z_{c-1}/Z_{c-3}} \cong Z_c/Z_{c-1} \cong Q/Z(Q).$$

Therefore $Z(Q)$, being cyclic, is of order p and so Q is an extra special p-group. Then $|Q| = |P/M| = p^{2s+1}$ for some natural number s ([GO, p. 204]), contradicting [M, 7.1].

CASE 6: $\mathrm{cl}(P) \leqq p+1$ *and* $M = P'$. Here $M = Z_{c-1}(P) = P'$. Set $Q = P/P_4$ (possibly $P_4 = 1$). Then (Q, Q') is an F2-pair with $Q' = P'/P_4 = M/P_4$. Also $\mathrm{cl}(Q) = 3$ and in particular Q' is abelian ([H, III. 2.11]). Let

$$|M : P_3| = |Q' : Q_3| = p^n \quad \text{and} \quad |P_3 : P_4| = |Q_3| = p^r.$$

By [M, 5.2, 2.3 and M1, 2.1] we know that $|P : M| = |Q : Q'| = p^{2n}$, $r \leqq n$, n is even and that Q/Q' and Q'/Q_3 are elementary abelian.

Let $x \in Q \setminus Q'$ and $y \in C_Q(x)$. We claim that $|\langle x, y, Q'\rangle/Q'| = p$. To see that write $x = aP_4$, $y = bP_4$ with $a, b \in P \setminus P_4$. As $[x, y] = 1$ we have that $[a, b] \in P_4$. Set $A = \langle a, b\rangle$ and $H = \langle a, b, P_4\rangle = P_4A$. By [H, III. 1.11] we obtain that $[A, A] \subseteq P_4$. We now use [H, III. 1.6, 1.10, 2.11] to observe that $H_i \subseteq P_{i+2}$ for all $i \geqq 2$. Indeed,

$$H_2 = [H, H] = [P_4A, H] = [P_4, H][A, H]$$
$$= [P_4, H][A, P_4][A, A] \subseteq P_4P_5P_4 \subseteq P_4;$$

so $H_2 \subseteq P_4$. Assume by induction that $H_i \subseteq P_{i+2}$ for some i. Then:

$$H_{i+1} = [H_i, H] \subseteq [P_{i+2}, H] \subseteq P_{i+3}$$

as needed. By assumption $P_{p+2} = 1$ and thus $H_p \subseteq P_{p+2} = 1$ so that $\mathrm{cl}(H) \leqq p-1$. In particular H is regular. By Lemma 2.1 $H/H \cap P'$ in an FW-complement and so [LP] implies that $H/H \cap P' \cong HP'/P'$ is cyclic. Denote images modulo P_4 by bars. Then

$$\overline{HP'}/\overline{P'} = \langle \bar{a}, \bar{b}\rangle \overline{P_4}\overline{P'}/\overline{P'} = \langle x, y\rangle Q'/Q'.$$

We conclude that $U = \langle x, y\rangle Q'/Q'$ is cyclic. But $U \subseteq Q/Q'$ which is elementary abelian so that $|U| = p$, as claimed.

As $x \notin Q'$, $\langle x, Q'\rangle/Q'$ itself has order p so that $U = \langle x, Q'\rangle/Q'$ and consequently $y \in \langle x, Q'\rangle$ for all $y \in C_Q(x)$ so $C_Q(x) \subseteq \langle x, Q'\rangle$ for all $x \in Q \setminus Q'$. Now

$$|\langle x, Q'\rangle| = p\,|Q'| = p\,|Q' : Q_3|\,|Q_3| = p^{n+r+1}$$

and on the other hand

$$|C_Q(x)| = |C_{Q/Q'}(xQ')| = |Q : Q'| = p^{2n},$$

as (Q, Q') is an F2-pair. It follows that $p^{2n} \leqq p^{n+r+1}$ and since $r \leqq n$ we have that either $n = r + 1$ or $n = r$. If $n = r + 1$ then

$$|\langle x, Q'\rangle| = p^{2n} = |C_Q(x)| \qquad \text{for all } x \in Q \setminus Q'$$

which implies that $C_Q(x) = \langle x, Q'\rangle$ for all $x \in Q \setminus Q'$. This means that Q' centralizes $\langle Q \setminus \Phi(Q)\rangle \subseteq \langle Q \setminus Q'\rangle$. Thus $Q' \subseteq Z(Q)$ and so $\mathrm{cl}(Q) = 2$, a contradiction.

Therefore $r = n$, $|Q| = p^{4n}$ and $|\langle x, Q'\rangle : C_Q(x)| = p$ for all $x \in Q \setminus Q'$. For each such x let $R(x) = Q' \cap C_Q(x)$. Then $C_Q(x) = \langle x, R(x)\rangle$ and $|Q' : R(x)| = p$. As $Q_3 \subseteq Z(Q)$ we have that $Q_3 \subseteq R(x)$. Thus each $R(x)$ is a maximal subgroup of Q' containing Q_3. The number, λ, of maximal subgroups of Q' containing Q_3 is equal to the number of maximal subgroups of Q'/Q_3 which is elementary abelian of order p^n. Hence

$$\lambda = \frac{p^n - 1}{p - 1}.$$

Consider now the

$$\mu = \frac{p^{2n} - 1}{p - 1}$$

subgroups of order p of Q/Q'. Each of these subgroups is of the form $\langle xQ'\rangle$ for some $x \in Q \setminus Q'$ and so each of these yields an $R(x)$. Set $\xi = \mu/\lambda = p^n + 1$. As the number of all the $R(x)$'s is no more than λ we get that there exists ξ distinct subgroups of order p, $\langle x_iQ'\rangle$, for some $x_i \in Q \setminus Q'$, $1 \leqq i \leqq \xi$, such that

$$R \equiv R(x_1) = R(x_2) = \cdots = R(x_\xi).$$

As $x_i \in C_Q(R(x_i))$ we get that $x_i \in C_Q(R) \setminus Q'$ for all $i = 1, 2, \ldots, \xi$. As Q' is abelian, $Q' \subseteq C_Q(R)$ and so $\langle x_iQ'\rangle$ for $i = 1, 2, \ldots, \xi$ are distinct subgroups of order p of $C_Q(R)/Q'$. Hence $|C_Q(R)/Q'| \geqq p^{n+1}$ because a p-group of order p^n or less has no more than $(p^n - 1)/(p - 1) < \xi$ subgroups of order p.

Finally, let $t \in R \setminus Q_3 \subseteq Q' \setminus Q_3$. Such t exists for otherwise $R = Q_3$ which means that $|Q' : Q_3| = p$ which implies that $n = 1$, contradicting the fact that n is even. Clearly $C_Q(R) \subseteq C_Q(t)$ and $Q' \subseteq C_Q(t)$. Thus

$$|C_Q(t)/Q'| \geqq |C_Q(R)/Q'| \geqq p^{n+1}.$$

On the other hand we use [M, 5.2] to obtain that (Q, Q_3) is also an F2-pair, $Q' = Z_2(Q)$ and $Q_3 = Z(Q)$. Since $t \in Q' \setminus Q_3 = Z_2(Q) \setminus Z(Q)$ we get that $tQ_3 \in Z(Q/Q_3)$ and so:

$$|C_Q(t)| = |C_{Q/Q_3}(tQ_3)| = |Q/Q_3| = p^{3n} \quad \text{and} \quad |Q : C_Q(t)| = p^{4n-3n} = p^n.$$

It follows that

$$|C_Q(t)/Q'| = \frac{|Q : Q'|}{|Q : C_Q(t)|} = p^{2n-n} = p^n,$$

a final contradiction.

To complete the proof we consider the last case:

CASE 7: P contains an abelian subgroup, S, of index p^2. Recall that $\mathrm{cl}(P) \geqq 3$. Now, $M \subseteq P'$ (by [M, 2.1]). If $S = M$ then $S = M = P'$ with $|P : P'| = p^2$ and so $(P/P_3, P'/P_3)$ is an F2-pair in which $P'/P_3 = (P/P_3)'$, $\mathrm{cl}(P/P_3) = 3$ and $|P/P_3 : P'/P_3| = p^2$. This contradicts [M, 5.2]. If $|P : M| = p^2$ then $M = P'$ and we get a contradiction as above. Thus $|S| > |M|$ and by [M, 7.11] $|P : M| \geqq p^4$. Let $x \in S \setminus M$. Then the F2 property of (P, M) implies that

$$|S| \leqq |C_P(x)| = |C_{P/M}(xM)| \leqq |P : M|.$$

So, $|M| \leqq |P : S| = p^2$. By case 4 $M \neq Z(P)$ and by [M, 2.1] $M = Z_i(P)$ for some i. If $|M| = p$ then $M = Z(P)$, a contradiction. Thus $|M| = p^2$ and so $M = Z_2(P)$. Also,

$$|S| = |C_P(x)| = |C_{P/M}(xM)| = |P : M| \qquad \text{for all } x \in S \setminus M.$$

We claim that $M \subseteq S$. To see that, let $C = C_P(M)$. Then $|P : C|$ divides $|\mathrm{Aut}(M)|_p = p$ and so $|P : C| = p$. It follows that

$$|P : S \cap C| = |P : C|\,|C : C \cap S| = |P : C|\,|CS : S| \leqq p^3.$$

Hence $|S \cap C| > |M|$. Now, let $y \in S \cap C \setminus M$. By the above, $S = C_P(y) \supseteq M$, as claimed.

From the last two paragraphs we conclude that $S/M \subseteq Z(P/M)$. If $S/M < Z(P/M)$ then $|P/M : Z(P/M)| = p$ and so $P/M = P/Z_2(P)$ would be abelian forcing $\mathrm{cl}(P) = 3$, a contradiction. Therefore $S/M = Z(P/M)$ and so $S = Z_3(P)$.

Set $X = P/Z(P)$. Then $(X, Z(X))$ is an F2-pair and [M, 2.2] implies that $S/M = Z_3(P)/Z_2(P) \cong Z_2(X)/Z(X)$ is elementary abelian. On the other hand $S/M \subseteq C_P(x)/M = C_P(x)/(C_P(x) \cap M)$ and [S, 1.4] implies that S/M is cyclic. It follows that $|S/M| = p$ and consequently $|P:M| = p^3$ contradicting [M, 7.1]. ■

REMARK ON THE PROOF OF PART 7 OF THEOREM 3. By part 4 of Theorem 3, part 7 holds for all odd primes and for $p = 2$ when $\mathrm{cl}(P) = 3$. Hence we may assume that $p = 2$ and $\mathrm{cl}(P) = 4$. Also we may assume that $M = P \cap K \neq Z(P)$ (by part 5 of Theorem 3) and that $P \neq P'$ (by [M1, 3.1] and [CM, 5.1]). It follows that $M = P_3 = Z_2(P)$. Set $Q = P/P_3$. The proof is now by a detailed investigation of P and Q. We feel that the complete proof is rather long and it would be inappropriate to include it here. The complete proof is available from the authors.

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